

Stationary states in single-well potentials under symmetric Lévy noises

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We discuss the existence of stationary states for subharmonic potentials $V(x) \propto |x|^c$, $c < 2$, under action of symmetric α -stable noises. We show analytically that the necessary condition for the existence of the steady state is $c > 2 - \alpha$. These states are characterized by heavy-tailed probability density functions which decay as $P(x) \propto x^{-(c+\alpha-1)}$ for $|x| \rightarrow \infty$, i.e. stationary states possess a heavier tail than the corresponding α -stable law. Monte Carlo simulations confirm the existence of such stationary states and the form of the tails of corresponding probability densities.

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I. INTRODUCTION

The dynamics of many physical systems can be reduced to a standard model of a “particle” (relevant coordinate x) under action of a deterministic force $f(x)$ and the noise force including the action of all neglected degrees of freedom. The mathematical tool for such a description is given by a Langevin equation, which in the overdamped limit typically takes the form

$$\dot{x}(t) = f(x) + \zeta(t). \quad (1)$$

In Eq. (1) $f(x)$ stands for the deterministic force, while $\zeta(t)$ is the stochastic force — noise — describing interactions of a test particle with its complex surrounding. Commonly, it is assumed that the stochastic force is of the white type, for example representing the particle’s collisions with molecules of the bath. Following Generalized Central Limit Theorem, a large number of independent collisions will lead to white noise of the stable type, i.e. either to Gaussian noise or to more general Lévy noise. The presence of fluctuations distributed according to Lévy laws have been observed in various situations in physics, chemistry or biology [1, 2], paleoclimatology [3] or economics [4]. The situations pertinent to heavy-tailed distributions appear in context of different models [5–8], and are analyzed in an increasing number of studies [9–23].

The action of white Lévy noise leads to increments of the stochastic process which are distributed according to α -stable Lévy type distributions. Symmetric Lévy

distributions $p_\alpha(x; \sigma)$ are characterized by their Fourier-transforms (characteristic functions of the distributions) $\phi(k) = \int_{-\infty}^{\infty} e^{ikx} p_\alpha(x; \sigma) dx$ being [24–26]

$$\phi(k) = \exp[-\sigma^\alpha |k|^\alpha]. \quad (2)$$

The parameter α (where $\alpha \in (0, 2]$) is the stability index of the distribution describing, for $\alpha < 2$, the asymptotic decay of its tails

$$p_\alpha(x) \propto |x|^{-(1+\alpha)}. \quad (3)$$

Finally, σ is the scale parameter which controls the overall distribution width. The Gaussian distribution corresponds to a special case of a Lévy stable law with $\alpha = 2$ when $\sqrt{2}\sigma$ is interpreted as the standard deviation of the distribution. In more general cases, for $\alpha < 2$, the variance of α -stable densities diverges. For $\alpha < 1$, the mean value of α -stable densities also does not exist.

The present work addresses properties of Lévy flights in external potentials. Theoretical descriptions of such systems is based on the Langevin equation and/or Fokker-Planck equation, which is of the fractional order [27]. The research performed here extends earlier studies [9, 19–21] where analysis of Lévy flights in harmonic and superharmonic potentials has been presented. The discussion conducted there did not account for the problem of existence of stationary states in potential less steep than parabolic, which is the main topic of the present work. This issue is addressed here by the use of analytical arguments and Monte Carlo simulations.

The model under discussion is presented in Section II. Section III discusses obtained results. The paper is closed with concluding remarks (Section IV).

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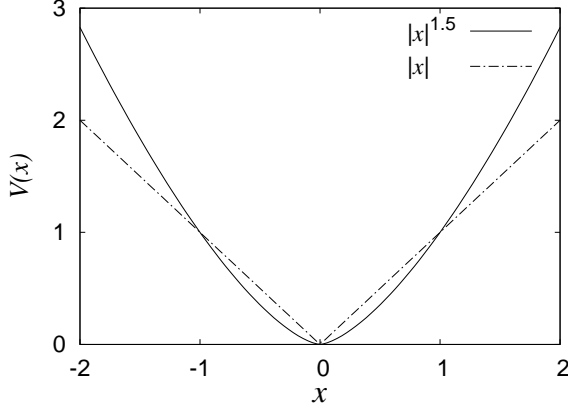


FIG. 1. Exemplary subharmonic potentials: $V(x) = |x|$ and $V(x) = |x|^{1.5}$ used for examination of the problem of existence of stationary states.

II. MODEL

An overdamped Brownian-Lévy particle moves in an external potential $V(x)$, therefore Eq. (1) takes the form

$$\dot{x}(t) = -V'(x) + \zeta(t), \quad (4)$$

where $V(x)$ represents a subharmonic single well potential $V(x) = |x|^c$ with $0 < c < 2$, see Fig. 1, and $\zeta(t)$ denotes a Lévy stable white noise process [9, 10, 15, 28, 29]. The position of a random walker can be calculated by means of the stochastic integration of Eq. (4) [25, 30]

$$\begin{aligned} x(t) &= x(0) - \int_0^t V'(x(s))ds + \int_0^t \zeta(s)ds \\ &= x(0) - \int_0^t V'(x(s))ds + L_\alpha(t). \end{aligned} \quad (5)$$

The integral $\int_0^t \zeta(s)ds \equiv L_\alpha(t)$ defines an α -stable Lévy process $L_\alpha(t)$ [9, 10, 15, 28, 31] which is driven by a Lévy stable noise $\zeta(t)$. Increments $\Delta L_\alpha(\Delta t) = L_\alpha(t + \Delta t) - L_\alpha(t)$ of the α -stable Lévy process are distributed according to the α -stable density with the stability index α . If the time step is set to Δt , the appropriate α -stable distribution is $p_\alpha(\Delta L_\alpha; \sigma(\Delta t)^{1/\alpha})$ [25, 26, 32, 33].

The equation (4) is associated with the following fractional Fokker-Planck equation (FFPE) [34–38]

$$\frac{\partial P(x, t)}{\partial t} = \left[\frac{\partial}{\partial x} V'(x, t) + \sigma^\alpha \frac{\partial^\alpha}{\partial |x|^\alpha} \right] P(x, t) \quad (6)$$

where the fractional (Riesz-Weyl) derivative is defined by the Fourier transform [9, 19, 20, 39] $\mathcal{F} \left[\frac{\partial^\alpha}{\partial |x|^\alpha} f(x) \right] = -|k|^\alpha \mathcal{F}[f(x)]$. The (space) fractional derivative in Eq. (6) describes long jumps which are distributed according to the α -stable density, see Eq. (2) and Refs. [1, 35, 37, 40]. The Langevin equation (4) provides a stochastic representation of the fractional Fokker-Planck

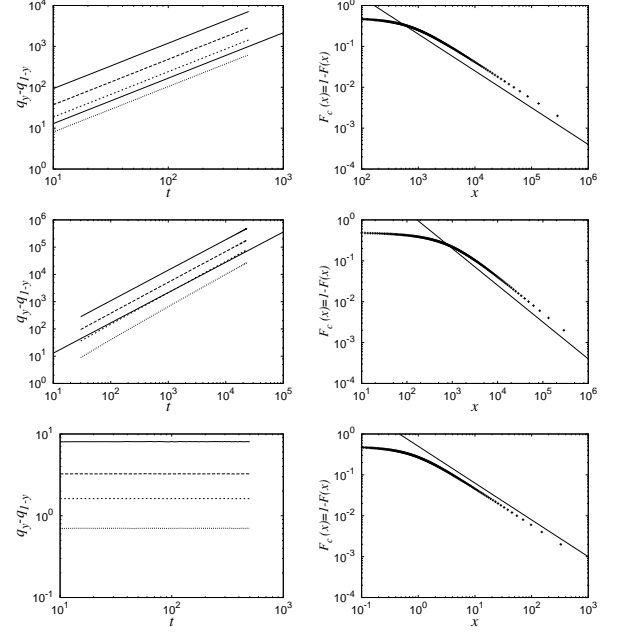


FIG. 2. Left column presents interquantile distance ($q_{0.9} - q_{0.1}$, $q_{0.8} - q_{0.2}$, $q_{0.7} - q_{0.3}$ and $q_{0.6} - q_{0.4}$, from top to bottom) as a function of time t . Solid lines present $t^{1/\alpha}$ scaling of the interquantile distance, which is observed in the force free case. Right column demonstrates complementary cumulative distributions at the end of simulation. Solid lines present $x^{-\alpha}$ decay. Various rows correspond to different potentials: $V(x) = 0$ (top panel), $V(x) = |x|$ (middle panel) and $V(x) = x^2/2$ (bottom panel). The stability index α and the scale parameter σ are set to $\alpha = 0.9$ and $\sigma = 1$ respectively.

equation (6). In other words, it describes an evolution of a single realization of the stochastic process $\{x(t)\}$. From ensemble of trajectories it is possible to analyze further properties of the system, in particular a probability density $P(x, t)$ of finding a random walker at time t in the neighborhood of x , see Eq. (6). More details on numerical scheme of integration of stochastic differential equations with respect to α -stable noises can be found in [14, 15, 25, 26, 32].

In the following sections properties of stationary probability distributions for single-well systems perturbed by symmetric Lévy noises are discussed. In the limiting cases of parabolic and quartic potentials the performed simulations corroborate earlier theoretical findings [9, 19–21, 41].

III. RESULTS

Only in the limited number of special cases it is possible to find stationary probability densities $P(x)$ analytically by solving Eq. (6). Therefore, the construction of stationary densities rely on numerical methods. In such a case, there are two frameworks possible: one option is

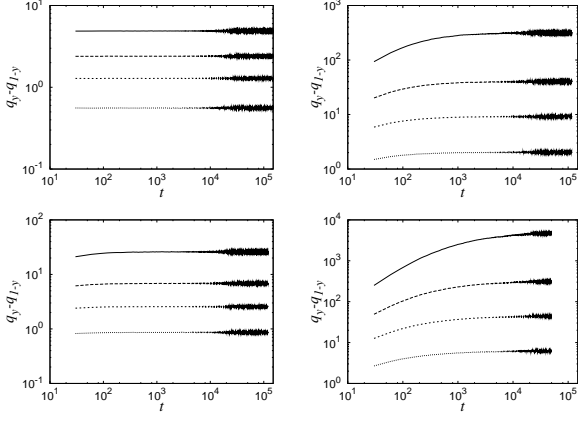


FIG. 3. Interquantile distance ($q_{0.9}-q_{0.1}$, $q_{0.8}-q_{0.2}$, $q_{0.7}-q_{0.3}$ and $q_{0.6}-q_{0.4}$, from top to bottom) as a function of time t for $V(x) = |x|^{1.5}$ with different values of the stability index $\alpha = \{1.5, 1.1\}$ (left column, from top to bottom) and $\alpha = \{0.9, 0.8\}$ (right column, from top to bottom). The scale parameter σ is set to $\sigma = 1$.

to discretize Eq. (6), see [19, 21, 42–45], alternatively one might use a Monte-Carlo method based on the simulation of the Langevin equation (4), see [25, 26, 30, 32, 46]. Here, we rely on Monte Carlo simulations. Such a choice is based on statistical properties of searched densities. As we proceed to show, for subharmonic potentials, if stationary states exist, the tail of the corresponding probability density function $P(x)$ is “fatter” than the tail of the corresponding α -stable density. In such a case confining the numerical solution of equation (6) to a finite interval may introduce uncontrollable errors. Therefore, in our simulations, we rely solely on stochastic representation of the fractional Fokker-Planck equation (6) which is provided by Eq. (4).

In the $\alpha = 2$ case (Gaussian noise) the stationary solution for Eq. (6), which in this case reduces to a “normal” Fokker-Planck equation, has the Boltzmann-Gibbs form

$$P(x) \propto \exp \left[-\frac{V(x)}{\sigma^2} \right] \quad (7)$$

and exists for any power-law potential $V(x)$ such that $\lim_{|x| \rightarrow \infty} V(x) = +\infty$. The situation drastically changes for Lévy noises with $\alpha < 2$. Stationary states always exist in potentials steeper than the parabolic one [19–21, 41]. However, as it will be shown this is no more the case for subharmonic potentials.

A. Parabolic and quartic potentials

There are two special cases of power-law potentials for which the stationary solution can be given in a closed form: the case of a parabolic potential (for any $0 < \alpha \leq 2$) and the case of a quartic potential for $\alpha = 1$ and

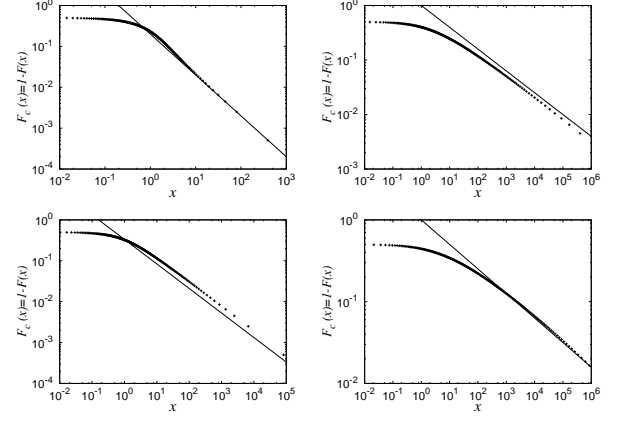


FIG. 4. Complementary cumulative distributions at the end of simulation for $V(x) = |x|^{1.5}$ with different values of the stability index $\alpha = \{1.5, 1.1\}$ (left column, from top to bottom) and $\alpha = \{0.9, 0.8\}$ (right column, from top to bottom). The scale parameter σ is set to $\sigma = 1$. Solid lines present $x^{-(\alpha-0.5)}$ decay predicted by Eq. (21).

$\alpha = 2$. These cases will serve as a benchmark for our further considerations.

In order to find stationary solution of the fractional Fokker-Planck equation (6) we rewrite it in the Fourier space

$$\frac{\partial \hat{P}(k, t)}{\partial t} = \hat{U}(k) \hat{P}(k, t) - \sigma^\alpha |k|^\alpha \hat{P}(k, t). \quad (8)$$

In Eq. (8) $\hat{U}(k)$ denotes the operator related to the Fourier representation of the potential $V(x)$ [21].

For the parabolic potential $V(x) = x^2/2$ the expression for $\hat{U}(k)$ reads $\hat{U}(k) = -k \frac{\partial}{\partial k}$. For symmetric α -stable noises, the stationary probability density fulfills [19, 20]

$$\frac{\partial \hat{P}(k)}{\partial k} = -\sigma^\alpha \text{sign} k |k|^{\alpha-1} \hat{P}(k). \quad (9)$$

The solution of Eq. (9) is

$$\hat{P}(k) = \exp \left[-\frac{\sigma^\alpha |k|^\alpha}{\alpha} \right], \quad (10)$$

i.e., the stationary solution is a symmetric Lévy distribution, see Eq. (2), characterized by a different value of the scale parameter than the noise in Eq. (4). The scale parameter of the stationary density is $\sigma/\alpha^{1/\alpha}$. For $\alpha < 2$, the variance of the stationary solution diverges and the parabolic potential is not sufficient to produce bounded states, i.e. states characterized by a finite variance [9, 19–21, 40].

For the quartic potential $V(x) = x^4/4$ the expression for $\hat{U}(k)$ reads $\hat{U}(k) = k \frac{\partial^3}{\partial k^3}$. The stationary density fulfills

$$\frac{\partial^3 \hat{P}(k)}{\partial k^3} = \sigma^\alpha \text{sign} k |k|^{\alpha-1} \hat{P}(k). \quad (11)$$

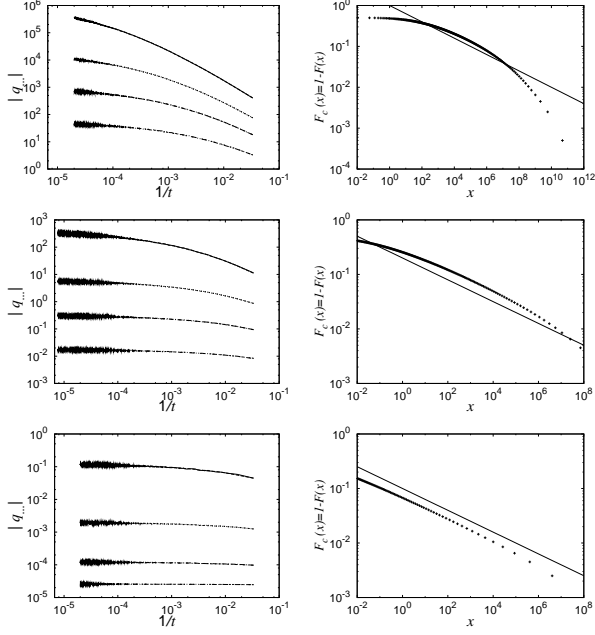


FIG. 5. Quantile lines q_y and $|q_{1-y}|$ ($y = \{0.9, 0.8, 0.7, 0.6\}$, from top to bottom) as a function of $1/t$ (left column) and complementary cumulative distributions at the end of simulation (right column) for $V(x) = |x|^{1.5}$ with the value of the stability index $\alpha = 0.7$ and the scale parameter σ : $\sigma = 1$ (top panel), $\sigma = 0.1$ (middle panel) and $\sigma = 0.01$ (bottom panel). In the right column, solid lines present $x^{-(\alpha-0.5)}$ decay predicted by Eq. (21).

For $\alpha = 1$, the solution of Eq. (11) reads [19, 20]

$$\hat{P}_{\alpha=1}(k) = \frac{2}{\sqrt{3}} \exp\left[-\frac{\sigma^{1/3}|k|}{2}\right] \cos\left[\frac{\sqrt{3}\sigma^{1/3}|k|}{2} - \frac{\pi}{6}\right]. \quad (12)$$

The formula for the corresponding stationary density $P(x)$ [19–21, 41] in the real space is

$$P_{\alpha=1}(x) = \frac{1}{\pi w [1 - (x/w)^2 + (x/w)^4]}, \quad (13)$$

where $w = \sigma^{1/3}$. The stationary solution (13) of Eq. (11), as well as the steady state (10) with $\alpha < 2$, is no longer of the Boltzmann-Gibbs type. This is typical for system driven by Lévy white noises with the stability index $\alpha < 2$ [47]. The stationary state, see Eq. (13), has the asymptotic power-law dependence, i.e. $P(x) \propto |x|^{-4}$. Furthermore, the stationary solution (13) is bimodal, with modal values located at $x = \pm\sigma^{1/3}/\sqrt{2}$ [19–21].

B. Subharmonic potentials

1. Analytical results.

Subharmonic potentials ($V(x) = |x|^c$ with $0 < c < 2$, see Fig. 1) interpolate between the force free case ($c = 0$)

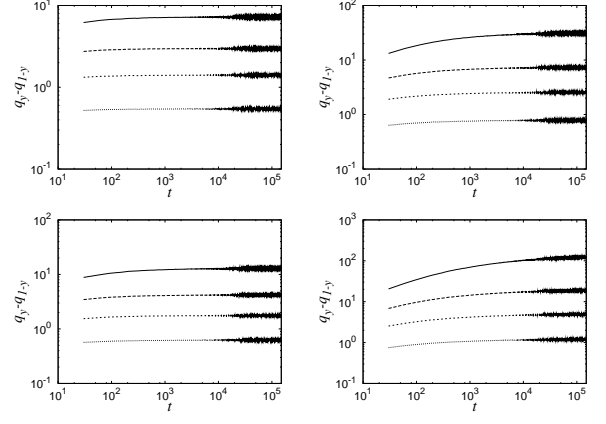


FIG. 6. Interquantile distance ($q_{0.9}-q_{0.1}$, $q_{0.8}-q_{0.2}$, $q_{0.7}-q_{0.3}$ and $q_{0.6}-q_{0.4}$, from top to bottom) as a function of time t for $V(x) = |x|$ with different values of the stability index $\alpha = \{1.7, 1.6\}$ (left column, from top to bottom) and $\alpha = \{1.5, 1.4\}$ (right column, from top to bottom). The scale parameter σ is set to $\sigma = 1$.

and the harmonic potential ($c = 2$). As it was indicated in the previous subsection for the harmonic potential, the stationary state is an α -stable Lévy type distribution, see Eqs. (2) and (10). In the force free case the stationary solutions do not exist. The time dependent solutions are just Lévy stable distributions with the growing scale parameter σ , i.e. $\sigma(t) = \sigma t^{1/\alpha}$. This observation suggests that for intermediate values of the exponent c ($0 < c < 2$) there should be a transition between the situation when the stationary state exists and the situation when the stationary state is absent.

It is possible to obtain a necessary condition for the existence of the stationary state using qualitative arguments based on the FFPE (6). If a stationary state for Eq. (6) exists, it satisfies

$$\left[\frac{d}{dx} V'(x) + \sigma^\alpha \frac{d^\alpha}{d|x|^\alpha} \right] P(x) = 0. \quad (14)$$

Assuming that tails of the distribution are given by a power-law, i.e. $P(x) \propto |x|^{-\nu}$ as $|x| \rightarrow \infty$, the first term in Eq. (14) behaves as

$$\frac{d}{dx} [c|x|^{c-1}P(x)] \simeq x^{c-2-\nu}. \quad (15)$$

The fractional derivative, see Eq. (14), is a nonlocal operator. The asymptotics of the fractional derivative of a probability density has a universal behavior independent of the particular form of this density. Indeed, its behavior for $x \rightarrow \infty$ is governed by the behavior of its Fourier-transform $-|k|^\alpha \hat{P}(k)$ for $k \rightarrow 0$. Due to the normalization condition $\hat{P}(k=0) = 1$. The inverse Fourier transform results in

$$\frac{d^\alpha}{d|x|^\alpha} P(x) \simeq x^{-1-\alpha}. \quad (16)$$

In order to satisfy Eq. (14), both terms (15) and (16) have to represent the same x -dependence. Consequently, we get $c - 2 - \nu = -1 - \alpha$ and

$$\nu = c + \alpha - 1. \quad (17)$$

Therefore a stationary state is asymptotically characterized by a power-law

$$P(x) \propto |x|^{-(c+\alpha-1)}, \quad (18)$$

i.e. by the same expression as for the superharmonic potentials [21]. The probability density function $P(x)$ has to be integrable, which corresponds to $\nu > 1$. Thus, the necessary condition for the existence of the steady state is

$$c > 2 - \alpha \quad (19)$$

or

$$\alpha > 2 - c. \quad (20)$$

For the parabolic potential ($c = 2$), the exponent ν reproduces the decay of the Lévy distribution ($\nu = 1 + \alpha$), see Eq. (10). For the quartic potential ($c = 4$) with $\alpha = 1$ we get $\nu = 4$, as follows from Eq. (13). The complementary cumulative density function ($F_c(x) = 1 - F(x) = 1 - \int_{-\infty}^x P(x')dx'$) is thus asymptotically characterized by a power-law decay

$$F_c(x) \propto x^{-(c+\alpha-2)}. \quad (21)$$

Having the condition (19), it is also possible to determine the dependence of the width of the stationary distribution $w(\sigma)$ on the noise strength σ . Assuming that $P(x) = w^{-1}f(x/w)$ can be expressed through a universal function $f(\xi)$ of a new, dimensionless, length variable $\xi = x/w$ (this assumption is reasonable due to the scale-free nature of the power-law potential), we change the length variable to $\xi = x/w$. For $V(x) = |x|^c$ (i.e. $V'(x) \propto c|x|^{c-1}$, $x \neq 0$), we get

$$w^{c-1}w^{-2} \frac{d}{d\xi} [c|\xi|^{c-1}f(\xi)] + \sigma^\alpha w^{-1-\alpha} \frac{d^\alpha}{d|\xi|^\alpha} f(\xi) = 0. \quad (22)$$

The universal rescaled equation for $f(\xi)$

$$\frac{d}{d\xi} [c|\xi|^{c-1}f(\xi)] + \frac{d^\alpha}{d|\xi|^\alpha} f(\xi) = 0 \quad (23)$$

can only be obtained if the width of distribution $w(\sigma)$ is the solution of the following equation

$$w^{c-3} = \sigma^\alpha w^{-1-\alpha}. \quad (24)$$

Condition (24) assures that prefactors of both terms in Eq. (22) scale similarly when σ is changed. Thus, from Eq. (24) one obtains

$$w(\sigma) \propto \sigma^{\alpha/(c+\alpha-2)}. \quad (25)$$

In the absence of the noise ($\sigma \rightarrow 0$) the solution $P(x)$ corresponds to the particle's position in the minimum of the potential, $P(x) = \delta(x)$, i.e. $w(\sigma) = 0$. For $c + \alpha < 2$ the exponent in Eq. (25) is negative, and the distribution's width $w(\sigma)$ diverges for $\sigma \rightarrow 0$ which corresponds to a non-physical situation. Consequently, no stationary state is possible. On the contrary, for $c + \alpha > 2$ the exponent in Eq. (25) is positive and the distribution width is the increasing function of the scale parameter σ . Therefore, the necessary condition of the existence of the steady state, see Eq. (19), is confirmed by the condition (25).

Note that the cases of parabolic and quartic potentials confirm Eq. (25). For the parabolic potential ($c = 2$) we get $w(\sigma) \propto \sigma$, see Eq. (10). In the case of quartic potential ($c = 4$) with $\alpha = 1$ we get $w(\sigma) \propto \sigma^{1/3}$, see Eq. (13). Moreover, it is also possible to discuss how does the relaxation time to the equilibrium depends on the scale parameter σ . To do this, we turn to Eq. (6) and rescale both the length ($\xi = x/w$) and the time variables ($\tau = t/\tau_0(\sigma)$), so that the rescaled solution to Eq. (6) is a universal function. The rescaling of t corresponds to

$$\tau_0(\sigma) \propto [w(\sigma)]^{3-c} = \sigma^{\alpha(3-c)/(c+\alpha-2)}. \quad (26)$$

It indicates that the dependence on the scale parameter σ for the stability index α close to $2 - c$ gets extremely sharp. In other words, when α approaches $2 - c$ (from above) the relaxation time grows and diverges at $\alpha = 2 - c$.

The condition (19) for the existence of the steady state can be corroborated by the following qualitative argument based on the decomposition of the noise into a “background” noise with finite variance, and “bursts” of arbitrary amplitude [48, 49].

Let us consider the discretized Langevin scheme with the time step of integration Δt and the characteristic amplitude of the Lévy jumps $\epsilon = \sigma(\Delta t)^{1/\alpha}$. Additionally, let us introduce a cutoff level Λ , so that only the jumps with the amplitude $a > \Lambda$ are considered as bursts. The overall situation is then considered as pertinent to a perturbation of the equilibrium distribution of particles in the potential $V(x)$ under the action of the background noise (i.e. the Boltzmann distribution in the potential $V(x)$ at the temperature $kT \simeq \Lambda$) by these distinct bursts. The width of this distribution W depends on the cutoff level Λ and on the type of the potential. Moreover, we assume that only very large bursts, with $a \gg \Lambda$, can be responsible for the absence of a stationary distribution. Therefore, the finite width of the background distribution can be neglected when considering the action of such bursts. The distribution of the burst amplitudes is then given by the asymptotic behavior of tails of the symmetric Lévy distribution

$$p(a) \simeq \frac{\epsilon^\alpha}{a^{1+\alpha}} = \frac{\sigma^\alpha \Delta t}{a^{1+\alpha}}. \quad (27)$$

A burst of amplitude a brings the particle to a position $x \approx a$, and the typical time necessary to return to the

interval of the characteristic width W is given by the solution of the ordinary equation of motion

$$\dot{x}(t) = -V'(x). \quad (28)$$

For $c \neq 2$, the return time $T(a)$ into the equilibrium domain, i.e. $|x| < W$, from the initial position $x = a$ is given by

$$T(a) = \frac{1}{c(2-c)} [a^{2-c} - W^{2-c}]. \quad (29)$$

For subharmonic potentials, $0 < c < 2$, the return time $T(a)$ is dominated by the initial position a . The leading contribution to $T(a)$ is provided by the first term of Eq. (29), i.e. $a^{2-c}/[c(2-c)]$. Therefore, instead of the return time to the equilibrium domain it is possible to consider the return time to the origin, which is finite and slightly larger than the return time to the characteristic domain.

It is reasonable to assume, that the steady state does not exist, if the time $T(a)$ to return from the point a is larger than the typical time between two bursts of the amplitude a or larger, and might exist in the opposite situation. The probability to have a burst of the amplitude a or larger is

$$P(a) = 2 \int_a^\infty p(a') da' \simeq \left(\frac{\epsilon}{a}\right)^\alpha, \quad (30)$$

so that the typical time between two such events is

$$T \simeq \Delta t \left(\frac{a}{\epsilon}\right)^\alpha. \quad (31)$$

Comparing T with $T(a)$ we conclude that the inequality $T(a) < T$ applies and the stationary states are possible for

$$c > 2 - \alpha. \quad (32)$$

For superharmonic potentials, $c > 2$, the situation is inverse to the situation for subharmonic potentials. The larger the initial displacement a is, the faster is the return to the W -domain. For the increasing initial displacement a , the typical return time essentially tends to a constant value $W^{2-c}/[c(c-2)]$, while the time between two subsequent bursts grows with their amplitude a . Comparing T with $T(a)$, we conclude that stationary states for superharmonic potentials exist for every value of the stability index α . The harmonic potential, $c = 2$, is the limiting case for which the return time shows not a power-law but a logarithmic dependence. In such a case, the very same arguments proofing existence of stationary states hold.

2. Numerical results.

Numerical results were obtained by Monte Carlo simulations of Eq. (4) with the time step of integration

$\Delta t = 10^{-2}$ and averaged over $N = 10^6$ realizations [14, 15, 25, 26, 32]. Initially, at time $t = 0$ a test particle was located at the origin, i.e. $x(0) = 0$. Due to the symmetry of the noise and of the potential, stationary states are symmetric with respect to the origin; the median and modal values of stationary densities are located at the origin.

Our test of stationarity is based on the quantiles of distributions. Quantiles of stationary distributions remain constant. Consequently, plotting quantile values as functions of time (quantile lines) gives us the key to analyze stationarity. More precisely, if the quantile lines are parallel to the abscissa, it means that the stationary state has been reached.

In order to verify performance of the test based on quantile lines we have constructed quantile lines for Lévy flights in the parabolic potential. In such a case the stationary state exists and is given by the α -stable density, see Eq. (10). Therefore, quantile lines should be parallel to the abscissa. Analogously, interquantile distance should not change over time. This behavior is indeed observed, e.g. in the bottom panel of Fig. 2.

Furthermore, Fig. 2 compares interquantile distance (left column) and complementary cumulative distributions $F_c(x, t) = 1 - F(x, t) = 1 - \int_{-\infty}^x P(x', t) dx'$ (right column) for the stability index $\alpha = 0.9$ and various potentials: $V(x) = 0$ (top panel), $V(x) = |x|$ (middle panel) and $V(x) = x^2/2$ (bottom panel). From these cases only the $V(x) = x^2/2$ case corresponds to the situation when stationary state exists. Top panel of Fig. 2 corresponds to the force free case in which no stationary density exists. In such a case, the interquantile distance grows like $t^{1/\alpha}$. The middle panel of Fig. 2 corresponds to the intermediate case, i.e. $V(x) = |x|$. Numerical simulations indicate that for $V(x) = |x|$ and $\alpha = 0.9$ no stationary state exists as well, what is coherent with analytical arguments given above. Note that in all three cases the decay of the probability densities follows the one of the noise, albeit on different reasons. For free motion, this decay follows the one of the noise due to the stability of the process. For the case $V(x) = |x|$, where according to analytical result the stationary state is absent, numerical simulations suggest that the tail of the distribution function is still dominated by the properties of the noise, and follows $F_c(x) \propto x^{-\alpha}$ pattern. Finally, for the parabolic potential it is essentially given by $F_c(x) \propto x^{-(c+\alpha-2)}$, i.e. again by $F_c(x) \propto x^{-\alpha}$ for $c = 2$.

Consecutive figures present results for subharmonic potentials. Figs. 3–5 present results for $V(x) = |x|^{1.5}$. Figs. 3–4 present: interquantile distance as a function of time t (Fig. 3) and complementary cumulative distributions ($F_c(x) = 1 - F(x)$) at the end of simulation (Fig. 4). Various panels present result for various values of the stability index α : $\alpha = \{1.5, 1.1\}$ (left column, from top to bottom) and $\alpha = \{0.9, 0.8\}$ (right column, from top to bottom). Finally, Fig. 5 presents quantile lines (left column) and complementary cumulative distributions (right column) for $\alpha = 0.7$ with various scale

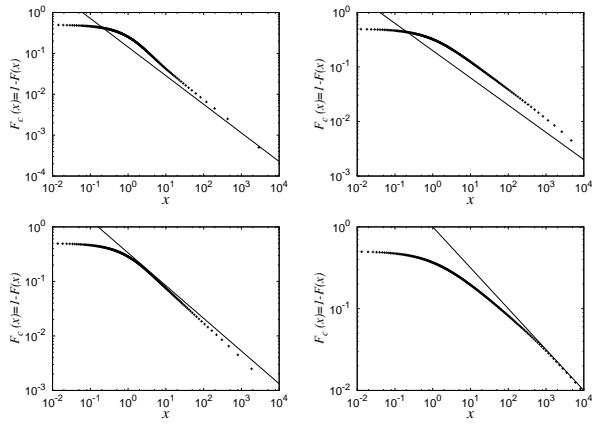


FIG. 7. Complementary cumulative distributions at the end of simulation for $V(x) = |x|$ with different values of the stability index $\alpha = \{1.7, 1.6\}$ (left column, from top to bottom) and $\alpha = \{1.5, 1.4\}$ (right column, from top to bottom). The scale parameter σ is set to $\sigma = 1$. Solid lines present $x^{-(\alpha-1)}$ decay predicted by Eq. (21).

parameters σ : $\sigma = 1$ (top panel), $\sigma = 0.1$ (middle panel) and $\sigma = 0.01$ (bottom panel).

Figures 6–8 present results for $V(x) = |x|$. Figs. 6–7 present: interquantile distance as a function of time t (Fig. 6) and complementary cumulative distributions ($F_c(x) = 1 - F(x)$) at the end of simulation (Fig. 7). Various panels present result for various values of the stability index α : $\alpha = \{1.7, 1.6\}$ (left column, from top to bottom) and $\alpha = \{1.5, 1.4\}$ (right column, from top to bottom). Finally, Fig. 8 presents quantile lines (left column) and complementary cumulative distributions (right column) for $\alpha = 1.3$ with various scale parameters σ : $\sigma = 1$ (top panel) and $\sigma = 0.1$ (bottom panel).

The stability index α in Figs. 3–8 is chosen in such a way that Eq. (19) holds true. Even in this case the numerical verification whether stationary states exist or do not exist, due to possible slow convergence to a stationary state, is not trivial, see discussion of Figs. 5 and 8 below.

The probability densities for the system described by Eqs. (4) and (6) are symmetric with respect to the origin. Therefore the quantile lines for q_y and $|q_{1-y}|$ overlap and are not distinguishable in left panels of Figs. 5 and 8.

For $V(x) = |x|^{1.5}$, Fig. 3 indicates that for sufficiently large values of the stability index α stationary densities exist. Although all the values of stability indices shown correspond to the domain where stationary states are predicted to exist, the convergence to these states becomes slower when α approaches $2 - c = 0.5$ for $V(x) = |x|^{1.5}$. Fig. 6 shows the very similar behavior for $V(x) = |x|$. Here again the relaxation time to the equilibrium gets larger when α approaches $2 - c = 1$.

Figures 4 and 7 present complementary cumulative distributions in the situation when Eq. (19) is valid. Therefore, according to Eq. (21), complementary cumulative

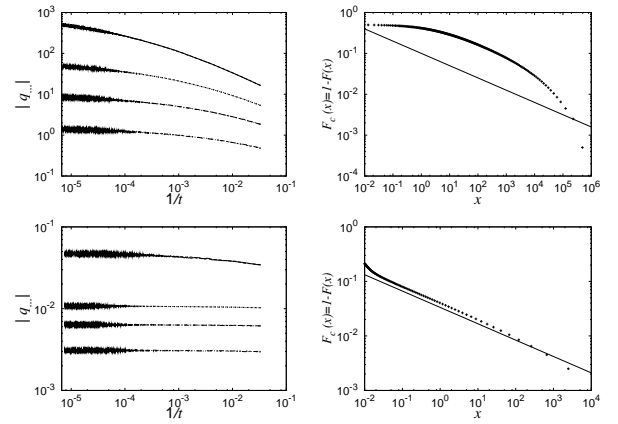


FIG. 8. Quantile lines q_y and q_{1-y} ($y = \{0.9, 0.8, 0.7, 0.6\}$, from top to bottom) as a function of $1/t$ (left column) and complementary cumulative distributions at the end of simulation (right column) for $V(x) = |x|$ with the value of the stability index $\alpha = 1.3$ and the scale parameter σ : $\sigma = 1$ (top panel) and $\sigma = 0.1$ (bottom panel). In the right column, solid lines present $x^{-(\alpha-1)}$ decay predicted by Eq. (21).

distributions should asymptotically decay as $x^{-(c+\alpha-2)}$. Indeed, in Figs. 4 and 7 the decay predicted by Eq. (21) is observed.

Figures 5 ($\alpha = 0.7$ and $V(x) = |x|^{1.5}$) and 8 ($\alpha = 1.3$ and $V(x) = |x|$) present quantile lines and complementary cumulative distributions for different values of the scale parameter σ . The quantile lines are plotted as functions of $1/t$ in order to elucidate convergence. Thus, in Fig. 5 the convergence of $q_{0.9}$ is not achieved over the time of simulation for $\sigma = 1$, but is approached for $\sigma = 0.01$. Correspondingly, the tail of the complementary cumulative density differs from theoretical predictions for $\sigma = 1$ and approaches its theoretical asymptotics for $\sigma = 0.01$. The same is true for the data in Fig. 8, where the convergence is achieved already for $\sigma = 0.1$. This indicates that the problem of the slow convergence to the stationary state has to be taken seriously.

IV. SUMMARY AND CONCLUSIONS

We considered the problem of existence of stationary states in power-law subharmonic potentials $V(x) = |x|^c$, $c < 2$, under the action of Lévy-stable noise characterized by the stability index α . The scaling analysis of the fractional Fokker-Planck equation leads us to the conclusion that such states exist if $\alpha > 2 - c$. This conclusion is corroborated by an alternative argument based on the decomposition of the Lévy noise. For subharmonic potentials, the probability density functions of stationary states are characterized by the asymptotic power-law decay $P(x) \propto |x|^{-\nu} = |x|^{-(c+\alpha-1)}$ for $|x| \rightarrow \infty$, which is slower than decay of the tails of the corresponding Lévy distributions. Monte Carlo simulations of the Langevin

equation confirm these analytical findings, at least for α not too close to $2 - c$. The convergence to the stationary state for α approaching $2 - c$ is very slow, which poses considerable numerical problems, since both long times and very small values of scaling parameter σ are required. However, for σ small enough the tendency to converge is still observable.

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